One more discussion of the replica trick: the examples of exact solutions

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A systematic replica field theory calculations are analysed using the examples of two particular one-dimensional "toy" random models with Gaussian disorder. Due to apparent simplicity of the model the replica trick calculations can be followed here step by step from the very beginning till the very end. In this way it can be easily demonstrated that *formally* at certain stage of the calculations the implementation of the standard replica program is just impossible. On the other hand, following the usual "doublethink" traditions of the replica calculations (i.e. closing eyes on the fact that certain suggestions used in the calculations contradict to each other) one can easily fulfil the programme till the very end to obtain physically sensible result for the entire free energy distribution function.

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I. INTRODUCTION

In recent years there is a renewed interest to the mathematical status of the replica method widely used in disordered systems during last four decades. For the calculation of thermodynamic quantities averaged over disorder parameters (e.g. average free energy) the method assumes, first, calculation of the averages of an integer n-th power of the partition function $Z(n)$, and second, analytic continuation of this function in the replica parameter n from integer to arbitrary non-integer values (and in particular, taking the limit $n \to 0$). Usually one is facing difficulties at both stages of this program. First of all, in realistic disordered systems the calculations of the replica partition function $Z(n)$ can be done only using some kind of approximations, and in this case the status of further analytic continuation in the replica parameter n becomes rather indefinite since the terms neglected at integer n could become essential at non-integer n (in particular the limit $n \to 0$) [1, 2]. The typical example of such type of trouble is provided by the classical Kardar's solution of $(1+1)$ directed polymers in random potential where due to the approximation used at the first stage of calculations (when the parameter n is still integer) the resulting free energy distribution function appears to be not positively defined [3, 4] (see also [5]). On the other hand, even in rare cases when the derivation of the replica partition function $Z(n)$ can be done exactly, further analytic continuation to non-integer n appears to be ambiguous. The classical example of this situation is provided by the Derrida's Random Energy Model (REM) in which the momenta $Z(n)$ growths as $\exp(n^2)$ at large n, and in this case there are many different distributions yielding the same values of $Z(n)$, but providing *different* values for the average free energy of the system [6]. Performing "direct" analytic continuation to non-integer n (just assuming that the parameter n in the obtained expression for $Z(n)$ can take arbitrary real values), one finds the so called replica symmetric (RS) solution which turns out to be correct at high temperatures, but which is apparently wrong (it provides negative entropy) in the low temperature (spin-glass) phase. In the case of REM the situation is sufficiently simple because here one can check what is right and what is wrong comparing with the available exact solution (which can be derived without replicas). Unfortunately in other systems the status of the results obtained by the replica method is much less clear.

In the case of the mean-field spin-glasses [7] the replica partition function also growths as $\exp(n^2)$ at large n, and its "direct" analytic continuation to non-integer n, as in REM, provides wrong RS solution in the low temperature spin-glass phase. Here the solution which is generally *believed* to be correct is obtained via the Parisi replica symmetry breaking (RSB) scheme (in the case of REM it reduces to the special case which is called one-step RSB), and it is derived in terms of a *heuristic* procedure and not as a proper analytic continuation from integer to non integer values of n of the replica partition function [8]. Recently the results obtained in terms of the RSB scheme has been confirmed by independent mathematically rigorous calculations (see [9] and references therein). Although no one seems to doubt now that the RSB heuristic procedure provide correct results, the problem is that until now no one was able to explain, *why* it provides correct results?

Presumably the most notable progress in the studies of the subtleties of the replica method has been achieved recently in the context of the random matrix theory, where the remarkable exact relation between replica partition functions and Painlevé transcendents has been proved [12–15]. One can also mention here recent exact replica solution for one-dimensional directed polymers in random potential, which (unlike all previous examples) *did not* involve an analytic continuation from integer to non-integer replica parameter n and where the results obtained were also expressed in terms of the Painlevé transcendents [10, 11].

In this paper I would like to consider two examples of the systematic replica field theory calculations using very

simple random systems for which, at first sight, every step of the replica program is under control. It turns out, however, that even in these extremely simple cases the derivation of the physical results inevitably requires the usual replica method "cheating" (the advantage of the simple system is that here one can easily see how it goes). Of course, it is not that present research explains something deep about the replica method. The aim of the paper is (once again) to turn the attention to the existing paradoxes, and to promote (once again) the idea that "something has to be done": it looks rather uncomfortable that on one hand we have extremely robust method, which in most of the cases works perfectly well, while on the other hand, we do not understand *why* it works.

The "toy" models considered in this paper are "extracted" from the one-dimensional directed polymers in a quenched random potential (for the detailed physical analysis of the obtained results see [16]). These systems describe an elastic string directed along the x-axis within an interval $[0, L]$ with displacements defined by the scalar field $\phi(x)$ having the elastic energy density proportional to $(\partial_x \phi)^2$. Randomness enters the problem through a disorder potential $V[\phi(x), x]$ competing against the elastic energy (see e.g. [17]). The problem then is defined by the Hamiltonian

$$H[\phi(x), V] = \int_0^L dx \Big\{ \frac{1}{2} \big[\partial_x \phi(x)\big]^2 + V[\phi(x), x] \Big\}; \tag{1}$$

The disorder potential $V[\phi(x), x]$ is Gaussian distributed with a zero mean $\overline{V(\phi, x)} = 0$ and a correlator

$$\overline{V(\phi, x) V(\phi', x')} = \delta(x - x') U(\phi - \phi'); \tag{2}$$

which is defined by a correlation function $U(\phi)$. The above equation implies that the random potential correlations are "translation invariant" in the ϕ direction depending only on the difference $(\phi - \phi')$.

General strategy of the replica calculations for this system is in the following. For the string with the zero boundary conditions at $x = 0$ the partition function of a given sample is

$$Z[V] = \int_{-\infty}^{+\infty} dy \int_{\phi(0)=0}^{\phi(L)=y} \mathcal{D}[\phi(x)] \; \mathrm{e}^{-\beta H[\phi, V]} \tag{3}$$

where the integration goes over all trajectories $\phi(x)$ staring at the origin, and β denotes the inverse temperature. On the other hand, the partition function is related to the total free energy $F[V]$ via

$$Z[V] = \exp(-\beta F[V]) \tag{4}$$

The free energy $F[V]$ is defined for a specific realization of the random potential V and thus represent a random variable. Let us take the n-th power of both sides of Eq.(4) and perform the averaging over the random potential V:

$$Z[n, L] = \overline{\Big(\exp(-\beta n F[V]) \Big)} \tag{5}$$

The quantity in the l.h.s of the above equation

$$Z[n, L] \equiv \overline{\Big(Z[V] \Big)^n} \tag{6}$$

is called the replica partition function, and it is defined originally for an arbitrary *integer* parameter n. Let us suppose that at large L the fluctuating free energy of the system scales with the system size as $F \propto L^\omega$, i.e. it is characterized by a single universal exponent ω. Redefining $F = f L^\omega$, we introduce a random quantity $f \sim 1$ which can be described by a distribution function $\mathcal{P}_L(f)$ (depending on the system size L). In this way, instead of eq.(5) we get the following general relation between the replica partition function $Z[n, L]$ and the distribution function of the (rescaled) free energy fluctuations $\mathcal{P}_L(f)$:

$$Z[n, L] \; = \; \int_{-\infty}^{+\infty} df \, \mathcal{P}_L(f) \; \mathrm{e}^{-\beta n L^\omega \, f} \tag{7}$$

The above equation is the bilateral Laplace transform of the function $\mathcal{P}_L(f)$, and at least formally it allows to restore this function in terms of the replica partition function $Z[n, L]$. In order to do so we have to compute $Z[n, L]$ for an *arbitrary* integer n and then (if the result would permit!) perform analytical continuation of this function from integer to arbitrary complex values of n. Introducing a new *complex* variable

$$s = \beta n L^\omega \tag{8}$$

and denoting

$$Z[\frac{s}{\beta L^{\omega}}, L] \equiv Z_L(s) \tag{9}$$

instead of eq.(7) we get

$$Z_L(s) \;=\; \int_{-\infty}^{+\infty} df\, \mathcal{P}_L(f)\, \mathrm{e}^{-s\, f} \tag{10}$$

According to this relation the distribution function $\mathcal{P}_L(f)$ can be reconstructed via the inverse Laplace transform

$$\mathcal{P}_L(f) \;=\; \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i}\, Z_L(s)\, \mathrm{e}^{sf}, \tag{11}$$

where the integration goes over the contour parallel to the imaginary axis. Finally, provided there are exist a *finite* thermodynamic limit function

$$\lim_{L\to\infty} Z_L(s) \;\equiv\; Z_*(s) \tag{12}$$

we can find the distribution function

$$\mathcal{P}_*(f) \;=\; \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i}\, Z_*(s)\, \mathrm{e}^{sf}, \tag{13}$$

which would describe the statistics of the rescaled free energy fluctuations f in the infinite system. The above equation defining $\mathcal{P}_*(f)$ contains no parameters and hence is expected to be universal. Note also that according to the relation $s = \beta n L^{\omega}$ in the thermodynamic limit $L \to \infty$ the relevant values of the replica parameter $n \sim L^{-\omega} \to 0$, which explains why the two limits $L \to \infty$ and $n \to 0$ do not commute [4].

Usually the problem formulated in eqs.(1)-(2) is studied in the context of the short-range correlated disorder potential, i.e., for a rapidly decaying function $U(\phi \to \infty) \to 0$ (which for simplicity is often replaced by the δ-function) In this case the free energy fluctuations scale as $L^{1/3}$ [3, 10, 11, 18–20]. On the other hand, if we would like to study the statistics of *small* string displacements we could develop the random potential $V[\phi, x]$ in powers of $\phi \ll 1$ keeping the first two terms only:

$$V[\phi, x] \simeq V_0(x) + V_1(x)\, \phi \tag{14}$$

where $V_0(x)$ and $V_1(x)$ are the Gaussian uncorrelated random parameters with the zero mean, $\overline{V_0(x)} = \overline{V_1(x)} = 0$, and the correlators

$$\begin{aligned} \overline{V_0(x)V_0(x')} \;&=\; v\, \delta(x - x') \\ \overline{V_1(x)V_1(x')} \;&=\; u\, \delta(x - x') \\ \overline{V_1(x)V_0(x')} \;&=\; 0 \end{aligned} \tag{15}$$

which implies that unlike eq.(2), the random potential correlator is *not* translation invariant:

$$\overline{V(\phi, x)V(\phi', x')} = \delta(x - x')\big[v \;+\; u\phi\phi'\big] \tag{16}$$

In this way, instead of eq.(1) we arrive to the Hamiltonian

$$H[\phi(x), V] = \int_0^L dx \Big\{ \frac{1}{2}\big[\partial_x \phi(x)\big]^2 + V_1(x)\phi(x) + V_0(x) \Big\}; \tag{17}$$

Now, lifting the requirement $\phi \ll 1$, we are getting simple Gaussian random force model to be studied in this paper by the replica method (Section II).

The thermodynamic limit, $L \to \infty$, of this system has been studied earlier [21]. Here I would like to concentrate on the thechnical details of the calculations. Due to apperent simplicity of the model the replica trick calculations can be followed here step by step from the very begining till the very end. In this way it can be easily demonstrated that

formally at sertain stage of the calculations the implementation of the replica program (as it is declared at the very begining) is just impossible. On the other hand, following the usual "doublethink" traditions of the replica calculations in disordered systems (i.e. closing eyes on the fact that certain suggestions used in the calculations contradict to each other) one can easily fulfill the programme till the very end to obtain very nice and physically sensible result for the free energy distribution function. Moreover, in this particular case we can be sure that the result obtained in this way is indeed *correct*, as it can also be derived via direct calculations without replicas [16]. It turns out that regardless of the apparent simplicity of the model, its free energy distribution function (which will be derived here for an arbitrary finite system size L) is rather non-trivial.

In Section III we consider slightly modified version of the above "toy" system. Namely, instead of developing the random potential itself, eq.(14), one can consider the development of its *correlation function* $U(\phi)$, eq.(2). Again, keeping the first two terms only, one gets

$$U(\phi) \;\simeq\; v - \frac{1}{2}u\phi^2 \tag{18}$$

(which, unlike eq.(16), would preserve the translation invariance of the random potential correlations). It turns out that this, seemingly rather innocent modification provokes quite dramatic consequences. As before, we lift the requirement $\phi \ll 1$ and accept the above "truncated" correlation function, eq.(18), as valid for the whole range of the scalar fields $\phi(x)$. Then, in the result of the standard replica calculations (which again inevitably involve the same cheeting as in the case of the previous model) we find that the corresponding free energy distribution function is *not positively defined*, which, of course, makes no physical sense. The point of this little methodological (and pedagogical) exercise is to demonstrate that it is not the replicas which are always responsible for all the troubles in the disordered systems world. In fact, the result of the replica calculations honestly reproduces the pathological nature of the original model itself: due to approximation, eq.(18), we obtain the replica theory which does not correspond to any physical system. One can show that there exists no (positively defined) Gaussian distribution function of the random potentials $V[\phi, x]$ which would provide the correlation function, eqs.(2), in the "parabolic" form, eq.(18). Moreover, the proper definition of the disorder potential is subject to important general constraints [22] regarding the shape of the correlation function $U(\phi)$, and neglection of these constraints may lead to unphysical results.

Indeed, consider a random potential $V(\phi)$ and its Fourier representation $\tilde{V}(p) = \int d\phi\; V(\phi)\exp(-ip\phi)$. Then the Gaussian distribution function of the random function $\tilde{V}(p)$ has the form

$$\mathcal{P}[\tilde{V}(p)] = P_0 \exp\Big(-\int \frac{dp}{2\pi}\,\frac{|\tilde{V}(p)|^2}{2G(p)}\Big); \tag{19}$$

where the *positive* function $G(p)$ is related with the correlation function $U(\phi)$ via

$$U(\phi) = \int \frac{dp}{2\pi}\, G(p)\, \exp(ip\phi). \tag{20}$$

Expanding both sides of the above relation in powers of ϕ,

$$U(0) + \sum_{k=1}^{\infty} \frac{1}{(2k)!} U^{(2k)}(0)\phi^{2k} = \int \frac{dp}{2\pi}\, G(p) + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!}\Big(\int \frac{dp}{2\pi} G(p)\, p^{2k}\Big)\phi^{2k}, \tag{21}$$

we can compare coefficients: the $2k$-th derivative of $U(\phi)$ in the origin relates to the integral $\int dpG(p)p^{2k}$ which is a positive quantity. Hence, we have to be careful in our choice of the correlator $U(\phi)$: if we truncate the expansion of $U(\phi)$ beyond some k^*, such that $U^{(2k)}(0) = 0$ for $k \geq k^*$, we impose the condition

$$\int dpG(p)p^{2k} = 0 \quad \text{for } k \geq k^*, \tag{22}$$

which cannot be satisfied for a positively defined $G(p)$. Obviously, choosing the correlator $U(\phi)$, in the "parabolic" form, eq.(18) is in severe conflict with this constraint. If nevertheless, we would substitute such a parabolic correlator into the replica Hamiltonian and perform all the standard calculations, first we would see no apparent indications telling that something went wrong. Moreover, one can easily calculate the average free energy of this system to discover that it perfectly coincides with the one of the (physically consistent) random force model, eq.(17). The trouble appears in the calculation of more specific quantities. For example, the second cumulant of free energy fluctuations turns out to be negative which makes no physical sense. Including the next term $\propto \phi^4$ in the correlator's expansion can cure this problem, however, an inconsistency then shows up in the next order cumulants, etc.

The paper is organized as follows. In Section II we perform detailed analycis of the systematic replica field theory calculations for the random force model, eq.(17), and present the results for its free energy distribution function. In Section III similar solution is considered for the "parabolic" directed polymer problem described by the correlation function, eq.(18), providing not positively defined free energy distribution function. Finally, the subtleties of the replica method are discussed in Section IV.

II. RANDOM FORCE MODEL

Explicitly, the replica partition function, Eq.(6), of the system described by the Hamiltonian, Eq.(17), is

$$Z(n,L) = \prod_{a=1}^{n} \int_{-\infty}^{+\infty} dy_a \int_{\phi_a(0)=0}^{\phi_a(L)=y_a} \mathcal{D}\phi_a(x)\, \overline{\exp\left[-\beta \int_0^L dx \sum_{a=1}^{n} \{ \frac{1}{2} \big[\partial_x \phi_a(x)\big]^2 + V_1(x)\phi_a(x) + V_0(x)\}\right]} \tag{23}$$

Since the random parameters $V_1(x)$ and $V_0(x)$ have Gaussian distribution with correlations defined in eq.(15), the disorder average $\overline{(...)}$ in the above equation is very simple:

$$\overline{\exp\left[-\beta \int_0^L dx \sum_{a=1}^{n} \big\{V_1(x)\phi_a(x) + V_0(x)\big\}\right]} \;=\; \exp\left[\frac{1}{2}\beta^2 \int_0^L dx \sum_{a,b=1}^{n} \big\{u\phi_a(x)\phi_b(x) + v\big\}\right] \tag{24}$$

Thus, the replica partition function, Eq.(23), can be represented in the following form

$$Z(n,L) \;=\; \mathrm{e}^{\frac{1}{2}\beta^2 n^2 v L} \prod_{a=1}^{n} \int_{-\infty}^{+\infty} dy_a \; \Psi\big[\mathbf{y}; L\big] \tag{25}$$

where the "wave function"

$$\Psi\big[\mathbf{y}; L\big] \;=\; \int_{\phi_a(0)=0}^{\phi_a(L)=y_a} \mathcal{D}\phi_a(x) \; \exp\Big[-\beta H_n[\boldsymbol{\phi}]\Big] \tag{26}$$

is defined by n-component scalar fields replica Hamiltonian

$$\begin{aligned} H_n[\boldsymbol{\phi}] \;&=\; \frac{1}{2}\int_0^L dx \left(\sum_{a=1}^{n} \big[\partial_x \phi_a(x)\big]^2 - \beta u \sum_{a,b}^{n} \phi_a(x)\phi_b(x) \right) \\ &=\; -\frac{1}{2}\int_0^L dx \sum_{a,b=1}^{n} \phi_a(x)\, U_{ab}\, \phi_b(x) \end{aligned} \tag{27}$$

The matrix

$$U_{ab} = \; \partial_x^2\, \delta_{ab} \;+\; \beta u \tag{28}$$

can be easily diagonalized. It has $(n-1)$-degenerate eigenvalue

$$\lambda_1 \;=\; \partial_x^2 \tag{29}$$

with $(n-1)$ orthonormal eigenvectors ξ_i^a such that

$$\sum_{a=1}^{n} \xi_i^a = 0 \qquad (i = 1, ..., n-1) \tag{30}$$

and one non-degenerate eigenvalue

$$\lambda_2 \;=\; \partial_x^2 \;+\; \beta n u \tag{31}$$

with the eigenvector

$$\xi_n^a = 1/\sqrt{n} \tag{32}$$

Thus, the matrix U_{ab}, Eq.(28), is diagonalized by the orthonormal transformation which is defined by the $(n \times n)$ matrix ξ_i^a (such that $\sum_{a=1}^n \xi_i^a \xi_j^a = \delta_{ij}$ and $\sum_{i=1}^n \xi_i^a \xi_i^b = \delta_{ab}$). In terms of new fields

$$\varphi_i(x) \;=\; \sum_{a=1}^{n} \xi_i^a \, \phi_a \tag{33}$$

the Hamiltonian, Eq.(27), takes the form

$$H_n[\boldsymbol{\varphi}] \;=\; \frac{1}{2}\int_0^L dx \sum_{i=1}^{n-1} \big[\partial_x \varphi_i(x)\big]^2 + \frac{1}{2}\int_0^L dx \Big\{ \big[\partial_x \varphi_n(x)\big]^2 - \beta n u \varphi_n^2(x) \Big\} \tag{34}$$

with the boundary conditions

$$\varphi_i(L) \;\equiv\; r_i(\mathbf{y}) \;=\; \sum_{a=1}^{n} \xi_i^a \, y_a \tag{35}$$

Correspondingly, the wave function, eq.(26), factorizes into

$$\Psi\big[\mathbf{y}; L\big] \;=\; \left[\prod_{i=1}^{n-1} \Psi_0\big[r_i(\mathbf{y}); L\big]\right] \Psi_1\big[r_n(\mathbf{y}); L\big] \tag{36}$$

where (with the proper choice of the integration measure)

$$\Psi_0(r; L) \;=\; \int_{\varphi(0)=0}^{\varphi(L)=r} \mathcal{D}\varphi(x) \; \exp\!\Big(-\frac{1}{2}\beta \int_0^L dx \big[\partial_x \varphi(x)\big]^2\Big) \;=\; \sqrt{\frac{\beta}{2\pi L}} \; \exp\!\Big(-\frac{\beta}{2L} r^2\Big) \tag{37}$$

and

$$\Psi_1(r; L) \;=\; \int_{\varphi(0)=0}^{\varphi(L)=r} \mathcal{D}\varphi(x) \; \exp\!\Big(-\frac{1}{2}\int_0^L dx \big[\beta\big(\partial_x \varphi(x)\big)^2 - \beta^2 n u \varphi^2(x)\big]\Big) \tag{38}$$

According to the above definition the wave function $\Psi_1(r,t)$ satisfies the imaginary-time Schrödinger equation

$$\partial_t \Psi(r;t) \;=\; \frac{1}{2\beta}\partial_r^2 \Psi_1(r;t) \;+\; \frac{1}{2}\,\beta^2 n u \, r^2 \, \Psi_1(r;t) \tag{39}$$

with the initial condition

$$\Psi_1(r; t=0) \;=\; \delta(r) \tag{40}$$

Eq.(39) describes the movement of a particle in the "reversed" parabolic potential. One can easily check that the solution of this equation satisfying the above initial condition is

$$\Psi_1(r;t) \;=\; b(t) \, \exp\!\Big(-\frac{1}{2} a(t) \, r^2\Big) \tag{41}$$

where

$$b(t) \;=\; \sqrt{\frac{\beta}{2\pi t}} \, \frac{(\lambda t^2)^{1/4}}{\sqrt{\sin(\sqrt{\lambda t^2})}} \tag{42}$$

and

$$a(t) \;=\; \beta \sqrt{\lambda} \, \frac{\cos(\sqrt{\lambda t^2})}{\sin(\sqrt{\lambda t^2})} \tag{43}$$

where we have introduced the parameter

$$\lambda \;=\; \beta n u \tag{44}$$

It should be stressed that the above solution exists provided

$$0 \;<\; \lambda t^2 \;<\; \frac{\pi^2}{4} \tag{45}$$

This restriction indicates that at a given value of the parameter λ the elastic string described by the partition function, eq.(38) (which contains negative mass!) goes to infinity at the *finite* time $t_c = \pi/(2\sqrt{\lambda})$. Coming back to the original random force problem one can reformulate the above restriction in the other way: for a given system size ("time") L the corresponding replica partition function $\overline{Z^n}$ is defined only if

$$n \;<\; n_c(L) \;=\; \frac{\pi^2}{4\beta u L^2} \tag{46}$$

At bigger values of n the replica partition function is simply not defined (it is formally divergent). Taking into account that the quantity $\pi^2/\big(4\beta u L^2\big)$ can be easily made less than one (e.g. by taking L sufficiently large), while the replica parameter n is still should be kept positive integer, we see that the above restriction becomes the fatal point for the whole replica scheme of the calculations. Of course, it is tempting to claim that since the replica n enter the partition function, eq.(38), in the form of the analytic parameter $\beta^2 n u$, we can analytically continue it to arbitrary real values in the interval $0 < n < n_c$, eq.(46). But the problem is that the expression for the whole replica partition function, eqs.(25), (36), requires that the replica parameter n *must* be still integer. In fact, the restriction, eq.(46), reflects simple mathematical reality: the replica partition function $\overline{Z^n}$ of the considered system is divergent at $n \geq n_c(L)$. In particular, at $L \geq \pi/\sqrt{4\beta u}$, the replica partition function does no exist (divergent) for *all* integers $n = 1, 2, ...$ In this situation (from the point of view of the common sense) the application of the replica method program for the system under consideration looks completely impossible. Nevertheless, in terms of the *modus vivendi* of the replica method this problem is overcome in a very simple way. It is in this point that the "doublethink" begins: wherever the replica n appears in a form of an analytic parameter it is immediately considered as real and belonging to the desired region ($n \to 0$ in spin glasses, or $0 < n < n_c$ in the present case), while *at the same time* wherever n can not made non-integer (in the summations or in the products) it is still considered as an integer (note that similar, although slightly more sophisticated trick is used in the replica symmetry breaking construction). Thus, we continue our calculations just plainly assuming that the condition, eq.(46), is satisfied.

Substituting, the solutions eq.(37), (41)-(43) into eq.(36) , we get

$$\Psi\big[\mathbf{y}; L\big] \;=\; \sqrt{\frac{\sqrt{\lambda L^2}}{\sin(\sqrt{\lambda L^2})}}\left(\frac{\beta}{2\pi L}\right)^{n/2} \exp\left[-\frac{\beta}{2L}\sum_{i=1}^{n-1} r_i^2\big(\mathbf{y}\big) - \frac{\beta}{2L}\frac{\sqrt{\lambda L^2}\cos(\sqrt{\lambda L^2})}{\sin(\sqrt{\lambda L^2})} r_n^2\big(\mathbf{y}\big)\right] \tag{47}$$

Using the relations, eqs.(35), (30) and (32), and taking into account that the matrix ξ_i^a is orthonormal, after some efforts in simple algebra we obtain the following result for the n-particle wave function, eq.(26):

$$\Psi\big[\mathbf{y}; L\big] \;=\; \sqrt{\frac{\sqrt{\lambda L^2}}{\sin(\sqrt{\lambda L^2})}}\left(\frac{\beta}{2\pi L}\right)^{n/2} \exp\left[-\frac{\beta}{2L}\sum_{a=1}^{n} y_a^2 \;-\; \frac{\beta}{2Ln}\left(\frac{\sqrt{\lambda L^2}\cos(\sqrt{\lambda L^2})}{\sin(\sqrt{\lambda L^2})} - 1\right)\Big(\sum_{a=1}^{n} y_a\Big)^2\right] \tag{48}$$

where it assumed that (*sic*!) $\lambda L^2 = \beta n u L^2 < \pi^2/4$ whatever the values of β, u and L are. Substituting this result into eq.(25) and performing simple Gaussian integration (which, taking into account orthogonality of the matrix ξ_i^a is easier to do in terms of the parameters r_i using expression, eq.(47)), for the replica partition function we finally get sufficiently simple result

$$Z(n, L) \;=\; \frac{1}{\sqrt{\cos(\sqrt{\beta n u L^2})}}\;\exp\Big[\frac{1}{2}\beta^2 n^2 v L\Big] \tag{49}$$

Next, for further implementation of the general program of the reconstruction of the free energy distribution function as it was described in the Introduction, eqs.(3)-(11), let us introduce parameter

$$w \;=\; \beta n u L^2 \tag{50}$$

which is confined in the interval $0 < w < \pi^2/4$. In terms of this parameter the general relation between the replica partition function $Z(n,L)$ and the free energy distribution function $P_L(F)$ (cf. eq.(5)),

$$Z(n,L) \;=\; \int_{-\infty}^{+\infty} dF\; P_L(F)\; \exp\bigl(-\beta n F\bigr) \tag{51}$$

takes the form

$$Z_L(w) \;=\; \int_{-\infty}^{+\infty} dF\; P_L(F)\; \exp\bigl(-\frac{w}{uL^2}F\bigr) \tag{52}$$

where

$$Z_L(w) \equiv Z\bigl(\frac{w}{\beta u L^2}, L\bigr) \;=\; \frac{1}{\sqrt{\cos(\sqrt{w})}}\; \exp\Bigl[\frac{v}{2u^2L^3}w^2\Bigr] \tag{53}$$

Rescaling the free energy, $F \;=\; uL^2 f$, for the probability distribution function $\mathcal{P}_L(f)$ of the random quantity f we get the following relation

$$Z_L(w) \;=\; \int_{-\infty}^{+\infty} df\; \mathcal{P}_L(f)\; \exp\bigl(-wf\bigr) \tag{54}$$

(where $\mathcal{P}_L(f) \;=\; uL^2 P_L\bigl(uL^2 f\bigr)$). Next, performing the analytic continuation of the function $Z_L(w)$, eq.(53), from the interval $0 < w < \pi^2/4$ to the complex half-plain, $\mathrm{Re}\{w\} < \pi^2/4$, (which is unambiguous operation) the free energy distribution function $\mathcal{P}_L(f)$ can be obtained via the inverse Laplace transform

$$\mathcal{P}_L(f) = \int_{-i\infty}^{+i\infty} \frac{dw}{2\pi i}\, Z_L(w)\, \exp(wf), \tag{55}$$

where the integration goes over the contour parallel to the imaginary axes such that $\mathrm{Re}\{w\} < \pi^2/4$.

In the thermodynamic limit, according to eq.(53),

$$\lim_{L\to\infty} Z_L(w) \equiv Z_*(w) \;=\; \frac{1}{\sqrt{\cos(\sqrt{w})}} \tag{56}$$

Thus, according to eq.(55), for the distribution function of the rescaled free energy fluctuations of the infinite system, $\mathcal{P}_*(f) \equiv \lim_{L\to\infty}\mathcal{P}_L(f)$, we obtain the following (universal) result [21]:

$$\mathcal{P}_*(f) = \int_{-i\infty}^{+i\infty} \frac{dw}{2\pi i}\, \frac{\exp(wf)}{\sqrt{\cos(\sqrt{w})}} \tag{57}$$

The overall form of this function is shown in Figure 1. It is interesting to note that this function is identically equal to zero at $f > 0$. This is easy to understand by making simple mathematical analysis of the integral in eq.(57). Indeed, since at $f > 0$ the function $[\cos(\sqrt{w})]^{-1/2}\exp(wf)$ quickly goes to zero at $w \to -\infty$, the contour of integration in the complex plane can be safely shifted to $-\infty$, which means that $\mathcal{P}_*(f < 0) \equiv 0$. The fact that in the thermodynamic limit the upper bound for the free energy of the system described by the Hamiltonian, eq.(17) is equal to zero can also be explained in terms of simple physical arguments. First we note that the typical value of the random constant term $\int dx V_0(x)$ scales as $L^{1/2}$, which means that its contribution to the rescaled free energy $f \sim F/L^2$ scales as $L^{-3/2}$ and vanishes in the thermodynamic limit $L \to \infty$. On the hand, since the contribution of the "trivial" configuration $\phi(x) = 0$ in the elastic and the random force terms of the Hamiltonian is equal to zero, any deviation from this configuration (due to the actions of the random force) can only reduce the energy. The asymptotic behavior of the function $\mathcal{P}_*(f)$ in the limits $f \to -\infty$ and $f \to -0$ can be easily estimated by the saddle-point integration to yield: $\mathcal{P}_*(f \to -\infty) \sim \exp\bigl(-\frac{\pi^2}{4}|f|\bigr)$ and $\mathcal{P}_*(f \to -0) \sim \exp\bigl(-\frac{1}{32|f|}\bigr)$.

At finite system size L, according to eqs.(53) and (55), the free energy distribution function is given by

$$\mathcal{P}_L(f) = \int_{-i\infty}^{+i\infty} \frac{dw}{2\pi i}\, \frac{1}{\sqrt{\cos(\sqrt{w})}}\; \exp\Bigl[\epsilon(L)w^2 + fw\Bigr] \tag{58}$$

where

$$\epsilon(L) \;=\; \frac{v}{2u^2L^3} \tag{59}$$

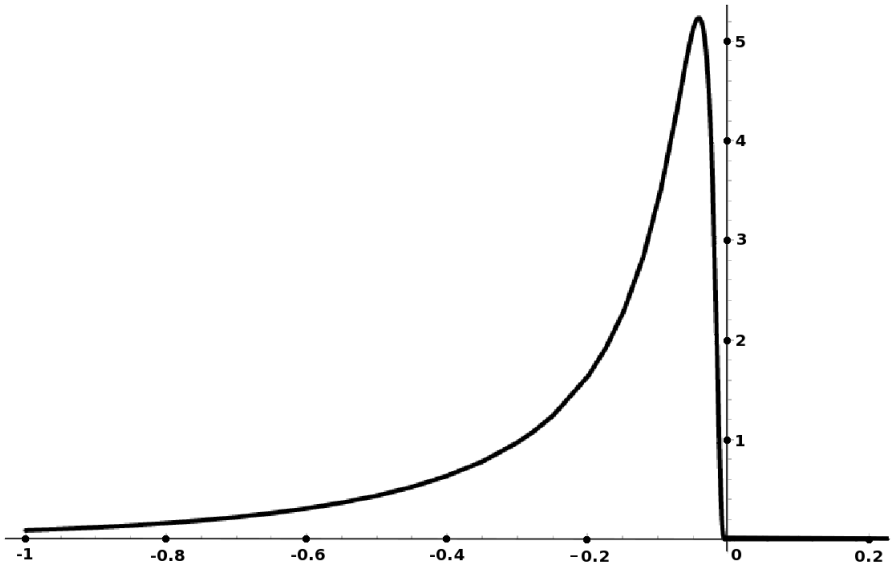

FIG. 1: The thermodynamic limit free energy distribution function $\mathcal{P}_*(f)$

The plot of this function for several values of the parameter $\epsilon(L)$ is shown in Figure 2. At small values of ϵ (large L) the function $\mathcal{P}_L(f)$ becomes close to the universal distribution function $\mathcal{P}_*(f)$ shown in Figure 1, while at large ϵ (when $v \gg u^2L^3$) $\mathcal{P}_L(f)$ becomes almost Gaussian, as in this case the free energy of the system is dominated by the Gaussian random constant term $\int dx V_0(x)$ of the Hamiltonian, eq.(17).

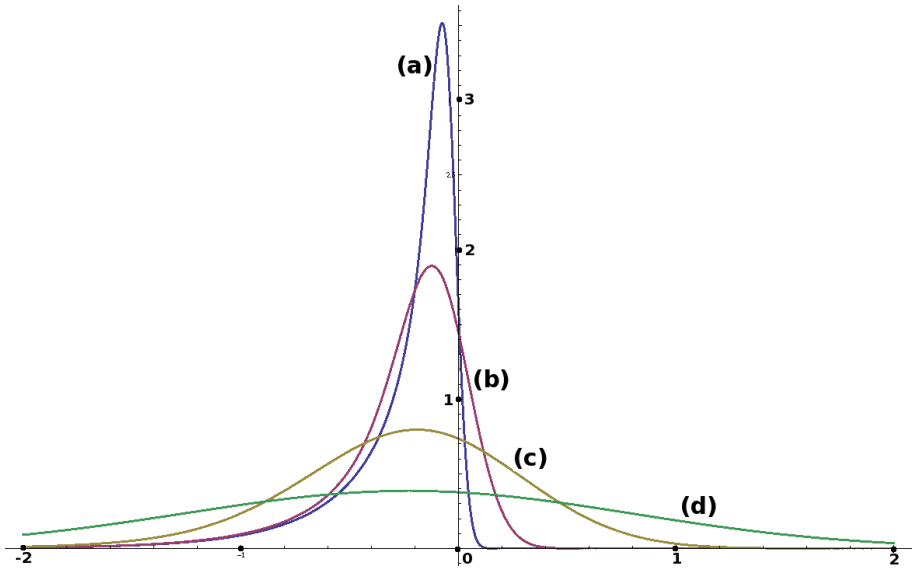

FIG. 2: Free energy distribution function $\mathcal{P}_L(f)$, eq.(58), for: (a) $\epsilon = 0.001$, (b) $\epsilon = 0.01$, (c) $\epsilon = 0.1$, (d) $\epsilon = 0.5$,

The result, eq.(58), constitutes the complete solution of the random force problem defined by the Hamiltonian, eq.(17). Could we call this solution "exact"? One one hand, the explicit cheating in the the derivation of the replica partition function, eq.(49), makes the status of the obtained result rather indefinite. On the other hand, it should be stressed that this kind free handling with the integer/non-integer status of the replica parameter n is just the routine trick in all replica field theory calculations in disordered systems (see e.g. [8, 23]). In other words, the results abtained in this way should be accompanied by the label "in the framework of the replica approach". Fortunately, in this particular case, due to Gaussian nature of the considered model the result for the replica partition function, eq.(49), (where $n \in \big(0, n_c(L)\big)$ is a *real* parameter) can be confirmed by independent calculation without the use of

replicas [16], which allows to claim that the above result, eq.(58), for the free energy distribution function is indeed exact.

III. DIRECTED POLYMERS WITH "PARABOLIC" CORRELATIONS OF THE RANDOM POTENTIAL

The replica partition function of directed polymers described by the Hamiltonian (1) with non-local parabolic correlations of the random potentials, Eq.(18), can also be represented in the form of eqs.(25)-(26) with the replica Hamiltonian

$$\tilde{H}_n[\boldsymbol{\phi}] \;=\; \frac{1}{2}\int_0^L dx\Biggl(\sum_{a=1}^{n}\bigl[\partial_x\phi_a(x)\bigr]^2+\frac{1}{2}\beta u\sum_{a,b}^{n}\Bigl[\phi_a(x)-\phi_b(x)\Bigr]^2\Biggr)$$

$$=\; -\frac{1}{2}\int_0^L dx\sum_{a,b=1}^{n}\phi_a(x)\,\tilde{U}_{ab}\,\phi_b(x) \tag{60}$$

with the matrix

$$\tilde{U}_{ab} = \Bigl(\partial_x^2 \;-\; \beta n u\Bigr)\delta_{ab} \;+\; \beta u \tag{61}$$

Following the same route as in the case of the random force model, eqs.(28)-(34), we note that the above matrix has $(n-1)$-degenerate eigenvalue $\tilde{\lambda}_1=\partial_x^2-\beta nu$ with $(n-1)$ orthonormal eigenvectors ξ_i^a constrained by the condition $\sum_{a=1}^{n}\xi_i^a=0$ $(i=1,...,n-1)$, and one non-degenerate eigenvalue $\tilde{\lambda}_2=\partial_x^2$ with the eigenvector $\xi_n^a=1/\sqrt{n}$. In terms of the new fields $\varphi_i(x)=\sum_{a=1}^{n}\xi_i^a\phi_a$, the replica Hamiltonian take the form (cf. eq.(34))

$$\tilde{H}_n[\boldsymbol{\varphi}] \;=\; \frac{1}{2}\int_0^L dx\sum_{i=1}^{n-1}\Bigl\{\bigl[\partial_x\varphi_i(x)\bigr]^2-\beta nu\phi_i^2(x)\Bigr\}+\frac{1}{2}\int_0^L dx\bigl[\partial_x\phi_n(x)\bigr]^2 \tag{62}$$

Similarly to the calculations of the previous section, eqs(36)-(49), for the replica partition function we get the following result (cf. eq.(49)):

$$\tilde{Z}(n,L) \;=\; \left[\frac{1}{\sqrt{\cos(\sqrt{\beta nuL^2})}}\right]^{(n-1)}\;\exp\Bigl[\frac{1}{2}\beta^2n^2vL\Bigr] \tag{63}$$

where, as in the case of the random force model, it is assumed that $\beta nuL^2<\pi^2/4$. In terms of the parameter $w=\beta nuL^2$ instead of eq.(53) we obtain

$$\tilde{Z}_{L,\beta}(w)=\sqrt{\cos(\sqrt{w})}\,\left(\sqrt{\cos(\sqrt{s})}\right)^{-\frac{w}{2\beta uL^2}}\exp\Bigl[\frac{v}{2u^2L^3}w^2\Bigr] \tag{64}$$

and the probability distribution function of the (rescaled) free energy fluctuation $\tilde{\mathcal{P}}_L(f)$ is given by the inverse Laplace transform (cf. eq.(55)):

$$\tilde{\mathcal{P}}_{L,\beta}(f)=\int_{-i\infty}^{+i\infty}\frac{dw}{2\pi i}\,\tilde{Z}_L(w)\,\exp(wf), \tag{65}$$

Performing (numerical) integration in the above equation one can easily find that this function is *not positively defined* for any values of the parameters $\epsilon=v/(2u^2L^3)$ and $\kappa=1/(2\beta uL^2)$. For example, in the zero temperature limit (when $\kappa\to 0$) the above equation reduces to (cf. eq.(58)):

$$\lim_{\beta\to\infty}\tilde{\mathcal{P}}_{L,\beta}(f)\;\equiv\;\tilde{\mathcal{P}}_L^*(f)\;=\;\int_{-i\infty}^{+i\infty}\frac{dw}{2\pi i}\,\sqrt{\cos(\sqrt{w})}\,\exp\Bigl[\epsilon(L)w^2+fw\Bigr] \tag{66}$$

The plot of this function for several values of the parameter $\epsilon(L)$ is shown in Figure 3. We see that unlike $\mathcal{P}_L(f)$ (Figure 2) of the random force model, the function $\tilde{\mathcal{P}}_L^*(f)$ even at small L (large ϵ) (when it is almost Gaussian) has always a kind of the negalive "kink" at sufficiently large f.

Unfortunately we are not able to confirm (or reject) the result, eq.(63), by independent calculations, as the replica theory defined by the Hamiltonian, eq.(60), does not corresponds to any physical system (see the discussion of this issue in the Introduction, eqs.(19)-(22)). On the other hand, the clear lesson which we can learn from the exercise considered in this Section, is that unlike the "honest cheating" with the status of the replica parameter n (discussed in previous Section), any *approximations* made at the stage of the replica calculations (the exact correlator $U(\phi)$, eq.(2), is replaced by its truncated expansion, eq.(18)) could be just fatal for the physical meaning of the obtained results.

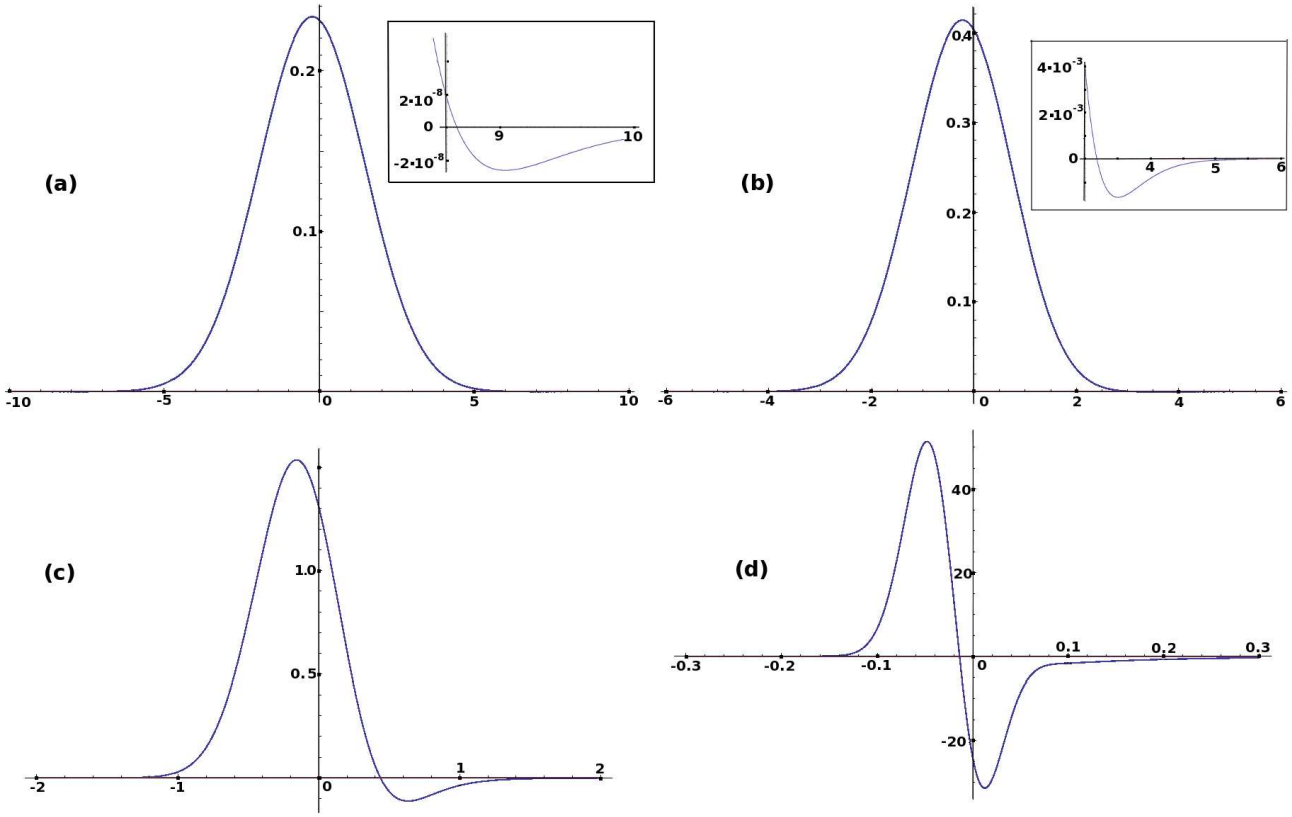

FIG. 3: Free energy "distribution function" $\tilde{\mathcal{P}}^*_L(f)$ as it given by eq.(66) for: (a) $\epsilon = 1.5$, (b) $\epsilon = 0.5$, (c) $\epsilon = 0.05$, (d) $\epsilon = 0.0005$,

IV. DISCUSSION

The standard program of the replica method is formulated as follows: first, for an arbitrary positive integer n we have to calculate the disorder average of the n-th power of the partition function, $\overline{Z^n} \equiv Z(n)$ which is expected to be an analytic function of the replica parameter n; second, we have to perform an analytic continuation of this function from integer for arbitrary real or complex values of n; and third, we have to take the limit $n \to 0$ (if we are interested in the average free energy only) or we have to perform an integration over complex n (if we are deriving the free energy distribution function). This third step is usually accompanied by taking the thermodynamic limit, which assumes that the system size L is taken to infinity. The prescription of the replica method indicates that the two limits, $n \to 0$ and $L \to \infty$, has to be taken simultaneously such that the product nL^ω (where an exponent ω defines the scaling of the free energy with the system size) is kept finite.

In fact, the whole experience of the replica calculations in disordered systems shows that except for trivial cases this program, as it is formulated above, is never followed! The typical illustration of the replicas *realpolitik* is provided by the studies in the mean-field spin glasses [7]. First of all, since the system is sufficiently complicated the computation of the replica partition function $Z(n)$ can be done here only in the saddle-point approximation and not exactly. This makes further analytic continuation to non-integer n somewhat doubtful because the neglected terms which are small

at *integer* n and large system size could become essential in the limit $n \to 0$. Moreover, it turns out that at large n the replica partition function growth as $\sim \exp(n^2)$ which means that its analytic continuation to non-integer n is ambiguous. If nevertheless we would just plainly take the limit $n \to 0$ in the obtained expression for $Z(n)$, we would get the so called replica symmetric (RS) solution which at low temperatures is unphysical since it reveals negative entropy and many other bad things. In view of the remarks made above, of course, this is not surprising. The strategy, which is called the replica symmetry breaking (RSB) scheme [8], and which is generally believed to provide correct results, is essentially different. In this scheme, all the above three steps, (computing $Z(n)$, analytic continuation in n and the limits $n \to 0$ and $L \to \infty$) are performed simultaneously! Similar (although slightly simplified) scheme works perfectly well also in the case of the Random Energy Model of spin glasses [6]. For both models the replica results are confirmed by independent mathematically rigorous calculations [9]. There are many other systems for which the results of the replica calcualtions, although can not be confirmed rigorously, are generally accepted to be correct [23].

To understand what is going on sometimes it is useful to consider an example of very simple system. Replica calculations for the model studied in this paper does not involve any kind of the RSB "magic". The model is so simple that initially one gets an illusion that every step of the calculations could be under good control. For instance, unlike the above example, the replica partition function here can be computed exactly for any (finite) system size, eq.(49). Nevertheless, proceeding with further steps of the replica method program one finds that either the attempt should be aborted, or one has to start cheating again. Indeed, the result, eq.(49), for the replica partition function turns out to be valid only for *finite* number of the integer points: $n < n_c = [\pi^2/(4\beta u L^2)]$, eq.(46), since at $n > n_c$ the quantity $Z(n)$ is not defined (it is formally divergent). Moreover, for sufficiently large system size, $L > L_c(n) = \pi/(2\sqrt{\beta u})$, the replica partition function $Z(n)$ is not defined for *all* positive integer n (including $n = 1$). In other words, in this situation the replica partition function of the considered system, defined as $\overline{Z^n}$ (where n is a positive integer) simply does not exist! But even if $L < L_c(n)$, so that $Z(n)$ is still defined at finite number of the integer points, it is evident that its analytic continuation for non-integer values of n is completely ambiguous. If nevertheless, one neglects all the above observations and accept the result, eq.(49), as valid for all *real* n in the interval $0 < n < n_c$, then everything becomes just fine. The analytic continuation of the function, eq.(49), from the finite interval $0 < n < n_c$ to the the complex half-plain, $\mathrm{Re}\{n\} < n_c$ is unambiguous, and in this way one obtains beautiful and *correct* results for the free energy distribution function (Figures 1 and 2). In this particular case one can be sure that obtained results are indeed correct as for the system under consideration the quantity $\overline{Z^n}$ can be computed directly for any *real* $n \in [0, n_c]$ [16].

All the experience of last decades convincingly demonstrate that with a few exceptions the replica method does give correct results, and this can not be explained by simple coincidences. We know very well *how* it works, and we do know that the replica calculations inevitably involves cheating. The question is then, *why* it works?

[1] J.J.M.Verbaarschot and M.R.Zirnbauer, *Critic of the replica trick*, J.Phys A: Math. Gen. **17**, 1093 (1985)

[2] M.R.Zirnbauer, *Another critic of the replica trick*, arXiv: cond-mat/9903338 (1999)

[3] M.Kardar, *Replica Bethe ansatz studies of two-dimensional interfaces with quenched random impurities*, Nucl. Phys. **B 290**, 582 (1987).

[4] E.Medina and M.Kardar, *Nonuniversality and analytical continuation in moments of directed polymers on hierarchical lattices*, J. Stat. Phys. **71**, 967 (1993).

[5] V.S.Dotsenko, L.B.Ioffe, V.B.Geshkenbein, S.E.Korshunov and G.Blatter, *Joint free energy distribution in the random directed polymer problem*, Phys. Rev. Lett. **100**, 050601 (2008)

[6] B. Derrida, *Random-energy model: An exactly solvable model of disordered systems*, Phys. Rev. B **24**, 2613 (1981).

[7] D. Sherrington and S. Kirkpatrick, *Solvable Model of a Spin-Glass*, Phys. Rev. Lett. **35**, 1792 (1975).

[8] M. Mezard, G. Parisi and M. A. Virasoro, *Spin Glass Theory and Beyond*, World Scientific (Singapore) 1987

[9] Adriano Barra, Aldo Di Biasio and Francesco Guerra, *Replica symmetry breaking in mean field spin glasses through Hamilton-Jacobi technique*, arXiv: 1003.5226 (2010)

[10] V.Dotsenko, EPL, **90**,20003 (2010); J.Stat.Mech. P07010 (2010)

[11] P.Calabrese, P. Le Doussal and A.Rosso EPL, **90**,20002 (2010).

[12] E.Kanzieper, *Replica field theories, Painlevé transcendents and exact correlation functions*, Phys. Rev. Lett. **89**, 250201 (2002).

[13] K.Splittorff and J.J.M.Verbaarschot, *Replica limit of the Toda lattice equation*, Phys. Rev. Lett. **90**, 041601 (2003).

[14] V.Al.Osipov and E.Kanzieper, *Are bosonic replicas faulty?*, Phys. Rev. Lett. **99**, 050602 (2007).

[15] E.Kanzieper, *Replica approach in random matrix theory*, arXiv: cond-mat/0903.3198v1 (2009).

[16] V.S.Dotsenko, V.B.Geshkenbein, D.A.Gorokhov and G.Blatter, *Free energy distribution function for randomly forced directed polymer* arXiv:1007.0852 (2010).

[17] T. Halpin-Healy and Y-C. Zhang, *Kinetic roughening phenomena, stochastic growth, directed polymers and all that. Aspects of multidisciplinary statistical mechanics*, Phys. Rep. **254**, 215 (1995).

[18] D.A. Huse and C.L. Henley, *Pinning and Roughening of Domain Walls in Ising Systems Due to Random Impurities*, Phys.

Rev. Lett. **54**, 2708 (1985);
[19] M. Kardar and Y-C. Zhang, *Scaling of Directed Polymers in Random Media*, Phys. Rev. Lett. **58**, 2087 (1987).
[20] D.A. Huse, C.L. Henley, and D.S. Fisher, Phys. Rev. Lett. **55**, 2924 (1985).
[21] D.A. Gorokhov and G. Blatter, *Exact Free Energy Distribution Function of a Randomly Forced Directed Polymer*, Phys. Rev. Lett. **82**, 2705 (1999).
[22] D.A. Gorokhov and G. Blatter, *Singularities of the renormalization-group flow for random elastic manifolds*, Phys. Rev. B **59**, 32 (1999).
[23] V.S. Dotsenko, *Introduction to the Replica Theory of Disordered Statistical Systems*, Cambridge University Press, (2001).